JOURNAL OF ALGEBRA 74, 466–493 (1982)

Approximation Complexes of Blowing-Up Rings

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Communicated by D. Buchsbaum

Received January 22, 1981

0. INTRODUCTION

The blowing-up rings we consider here are the symmetric and Rees algebras of ideals, and some of their fibers. To recall the meaning of these rings, let R be a commutative Noetherian ring and let I be an ideal. Then the Rees algebra of I is the form ring

$$R(I) = \bigoplus_{t \geqslant 0} I^t$$

that occurs in the process of blowing-up the variety associated to R along the subvariety defined by I. The symmetric algebra of I, $\mathrm{Sym}(I)$, also represents a blowing-up, but of a much looser structure; it has advantages over $R(I)$, however, in that properties of the ideal I are more readily reflected on its arithmetic properties.

There is a canonical surjection

$$\alpha: \mathrm{Sym}(I) \to R(I)$$

which we, heuristically, view as an approximation, whenever much is known

0021-8693/82/020466–28\$02.00/0

about one of these algebras. The expression of the comparison between these two algebras—that is, of the morphism α—is given in terms of the homology of certain complexes derived from a double Koszul complex associated to a generating set for the ideal. This was the approach pursued in [19, 20], where some of these complexes were introduced. Here other "approximation" complexes are brought in, and various relationships among them are examined. This framework allows not only for the determination of the kernel of α in qualitative terms, but also for a discussion of Cohen–Macaulay properties of symmetric algebras and associated graded rings, and, in some cases, of their multiplicities.

We shall now describe the contents of this paper; further comments are made at the appropriate sections.

The introductory Section 1 contains the construction of the double Koszul complex associated to an ideal, with an outline of its general properties. In Section 2, we introduce the approximation complexes and derive their basic relationships. Its main results are: (i) A close description of the kernel of the morphism α; it is sufficiently detailed to provide simple proofs of the earlier computational results of [17]. (ii) Along with the main result of [19], it exhibits large classes of ideals for which $\mathrm{Sym}(I) \cong R(I)$ and shows both $\mathrm{Sym}(I)$ and its fibre $\mathrm{Sym}(I/I^2)$ to be Cohen–Macaulay—often Gorenstein in the case of the latter (Theorem 2.6). The next section, Section 3, discusses a theme of [23], where an isomorphism

$$\mathrm{Sym}(I/I^2) \cong \mathrm{gr}_I(R) = \sum_{t \geqslant 0} I^t/I^{t+1}$$

between the fibres of $\mathrm{Sym}(I)$ and $R(I)$ is lifted to an isomorphism $\mathrm{Sym}(I) \cong R(I)$. The current context even permits a departure from Noetherian conditions. In Section 4 we give explicit formulas for the multiplicities of the graded rings arising from Section 2 (Corollary 4.2) and study the so-called syzygetic conditions, that is, the nature of the ordinary Koszul homology modules on a set of generators of the ideal. In Section 5 we examine the relationship of the approximation complexes to "sequencially" generated ideals—essentially ideals generated by various generalizations of the notion of regular sequence, e.g., d-sequences and relative regular sequences. Such notions have been studied by several authors (i.e., [5, 11, 21]), and at the risk of fuzzing further the picture, we bring in the notion of "proper" sequence. Besides new families of examples, the main results of this section are (i) the proof, for d-sequences, of acyclicity for all complexes (Theorem 5.6) and the consequent isomorphism $\mathrm{Sym}(I) \cong R(I)$; this isomorphism aspect was given earlier treatments in [12, 23]; (ii) the identification of a singularity locus of α for certain ideals generated by proper sequences (Theorems 5.8 and 5.9). Finally, we explain the (γ)-condition of [17].

1. The Double Koszul Complex

Throughout, unless otherwise specified, rings are Noetherian and modules are finitely generated. For terminology and basic properties of Noetherian rings, especially Cohen–Macaulay rings, we shall use [13, 15].

For a module M over the commutative ring R, we shall denote the exterior and symmetric algebras of M respectively by ΛM and $\mathrm{Sym}(M)$. Their components of degree t will be indicated by $\Lambda^t M$ and, in the case of the symmetric algebra, by either $\mathrm{Sym}_t(M)$ or $S_t(M)$.

Let

$$\begin{array}{ccc} G & \xrightarrow{\psi} & B \\ {\scriptstyle\phi}\downarrow & & \\ R & & \end{array}$$

be a diagram of R-modules and homomorphisms. We shall make the algebra $\Lambda G \otimes_R \mathrm{Sym}(B)$ into a double complex, with commuting differentials.

$$d_\phi = \partial\colon \Lambda^r G \otimes S_t(B) \to \Lambda^{r-1} G \otimes S_t(B)$$

is the usual Koszul complex with coefficients in $\mathrm{Sym}(B)$:

$$\partial(e_1 \wedge \cdots \wedge e_r \otimes w) = \sum (-1)^{r-i} e_1 \wedge \cdots \wedge \hat{e}_i \wedge \cdots \wedge e_r \otimes \phi(e_i) w.$$

The other differential is

$$d_\psi = \partial'\colon \Lambda^r G \otimes S_t(B) \to \Lambda^{r-1} G \otimes S_{t+1}(B),$$

$$\partial'(e_1 \wedge \cdots \wedge e_r \otimes w) = \sum (-1)^{r-i} (e_1 \wedge \cdots \wedge \hat{e}_i \wedge \cdots \wedge e_r \otimes \psi(e_i) \cdot w,$$

where $\psi(e_i) \cdot w$ is the product of $\psi(e_i)$ and w in $\mathrm{Sym}(B)$.

Straightforward verification shows that $\partial\partial' = \partial'\partial$. The resulting differential graded algebra will be denoted by $\mathscr{L} = \mathscr{L}(\phi, \psi)$.

Let $\mathbf{x} = \{x_1, \ldots, x_n\}$ be a generating set of the ideal I, and let $\phi\colon G = R^n \to R$ be the mapping defined by the matrix $[x_1, \ldots, x_n]$.

Definition. $\mathscr{L} = \mathscr{L}(\phi$, identity) will be called the *double Koszul complex* associated to the sequence $\mathbf{x}$.

To make the next observations apparent, we display a portion of the complex $\mathscr{L}$:

$$\begin{tikzcd}
\wedge^{r+2}(R^n)\otimes S_{t-1}(R^n) \arrow[r] \arrow[d] & \wedge^{r+1}(R^n)\otimes S_t(R^n) \arrow[r] \arrow[d] & \wedge^{r}(R^n)\otimes S_{t+1}(R^n) \arrow[d] \\
\wedge^{r+1}(R^n)\otimes S_{t-1}(R^n) \arrow[r] \arrow[d] & \wedge^{r}(R^n)\otimes S_t(R^n) \arrow[r, "\partial'"] \arrow[d, "\partial"'] & \wedge^{r-1}(R^n)\otimes S_{t+1}(R^n) \arrow[d] \\
\wedge^{r}(R^n)\otimes S_{t-1}(R^n) \arrow[r] & \wedge^{r-1}(R^n)\otimes S_t(R^n) \arrow[r] & A^{r-2}(R^n)\otimes S_{t+1}(R^n)
\end{tikzcd}$$

We note that the complex $\mathscr{L}(\partial)$ is simply the ordinary Koszul complex associated to the sequence $\mathbf{x}$, tensored with the polynomial ring $\mathrm{Sym}(R^n)=R[T_1,\dots,T_n]$. As for the "horizontal" complex $\mathscr{L}(\partial')$, it is also an ordinary Koszul complex, but constructed over a different sequence of elements of $R[T_1,\dots,T_n]$. Indeed, let $A=\mathrm{Sym}(R^n)$, $F=A^n=R^n\otimes_R A$ and let ψ be the composition

$$F \xrightarrow{id} R^n \otimes_R A \longrightarrow A,$$

where the last map is the natural multiplication into the augmentation ideal of the R-algebra A. $\mathscr{L}(\partial')$ is the corresponding Koszul complex, that is, $\mathscr{L}(\partial')$ is the ordinary complex associated to the sequence $\{T_1,\dots,T_n\}$ of elements of $R[T_1,\dots,T_n]$. Thus we have a grading of $\mathscr{L}(\partial')$ by the subcomplexes of R-modules, $\mathscr{L}=\sum \mathscr{L}_t$, where

$$\mathscr{L}_t=\sum_{r+s=t} \wedge^r(R^n)\otimes \mathrm{Sym}_s(R^n)$$

and the $\mathscr{L}_t$'s are exact for $t>0$. This observation plays a key role in the sequel.

2. Approximation Complexes

For a sequence of elements $\mathbf{x}=\{x_1,\dots,x_n\}$ generating the ideal $(\mathbf{x})=I$, denote by $\mathscr{K}=\mathscr{K}(\mathbf{x})$ the associated ordinary Koszul complex, and by $\mathscr{L}$ the double complex of Section 1. Since ∂ and ∂' commute, several complexes that live in $\mathscr{K}$ give rise to larger complexes when extended over $R[T_1,\dots,T_n]$ and the differential induced by ∂' is used. In this section we discuss some of these complexes—the so-called approximation complexes of the title—and their role in the comparison of several graded rings.

Denote by $Z(\mathscr{K})$ the cycles of $\mathscr{K}$ and by $B(\mathscr{K})$ its boundaries. Since the differential of $\mathscr{K}$ satisfies the property

$$\partial(\alpha\wedge\beta)=\alpha\wedge\partial(\beta)+(-1)^p\,\partial(\alpha)\wedge\beta, \qquad p=\text{degree of }\beta,$$

$Z(\mathscr{K})$ is actually an R-subalgebra of $\mathscr{K}$ and $B(\mathscr{K})$ is an ideal in $Z(\mathscr{K})$. Another ideal of $Z(\mathscr{K})$ consists of the elements in $Z(\mathscr{K})$ with coefficients in I, that is, $Z^*(\mathscr{K}) = Z(\mathscr{K}) \cap I \cdot \mathscr{K}$. Since $B(\mathscr{K}) \subset Z^*(\mathscr{K})$, we have a homomorphism of R-algebras (in fact, of R/I-algebras)

$$0 \to \delta(\mathbf{x}) \to H = Z(\mathscr{K})/B(\mathscr{K}) \to H^* = Z(\mathscr{K})/Z^*(\mathscr{K}) \to 0.$$

The basic complexes we will be dealing with are: $\mathscr{Z} = Z(\mathscr{K}) \otimes \mathrm{Sym}(R^n) \cong Z(\mathscr{K}) \otimes R[T_1,\dots, T_n]$, $\mathscr{B} = B(\mathscr{K}) \otimes \mathrm{Sym}(R^n) \cong B(\mathscr{K}) \otimes R[T_1,\dots, T_n]$ and $\mathscr{M} = H \otimes \mathrm{Sym}(R^n)$ with the differential induced by ∂'. Since the homology H of the complex $\mathscr{K}$ is annihilated by I, the last complex may also be written $\mathscr{M} = H \otimes \mathrm{Sym}_S(S^n) \cong H \otimes S[T_1,\dots, T_n]$, $S = R/I$.

Some of these complexes made their appearence in [19, 20], to which we shall refer for some of their properties that simply mimic those of the ordinary Koszul complexes. The theme of this section is the interpretation of the vanishing of the homology of the complexes $\mathscr{Z}$ and $\mathscr{M}$ at various dimensions. To a smaller extent, we shall look at the role of $\delta(\mathbf{x})$ in the acyclicity of the complex $\mathscr{M}$.

We make some remarks about $\mathscr{Z}$ and $\mathscr{M}$.

First, we claim that $H_0(\mathscr{Z}) = \mathrm{Sym}(I)$. Indeed, looking at the presentation of I provided by

$$0 \to Z_1(\mathscr{K}) \to R^n \to I \to 0$$

we obtain

$$Z_1(\mathscr{K}) \otimes \mathrm{Sym}(R^n) \to Z_0(\mathscr{K}) \otimes \mathrm{Sym}(R^n) = \mathrm{Sym}(R^n),$$

and the conclusion follows.

Similarly one shows that $H_0(\mathscr{M}) = \mathrm{Sym}_S(I/I^2)$. As a matter of notation, we shall always write $\mathrm{Sym}_S(I/I^2) = \mathrm{Sym}(I/I^2)$.

It is also easy to see that the algebra $H(\mathscr{Z})$ (resp. $H(\mathscr{M})$) is a finitely generated $\mathrm{Sym}(I)$ (resp. $\mathrm{Sym}(I/I^2)$)-module. As in [19], one shows that $H(\mathscr{Z})$ and $H(\mathscr{M})$ do not depend on the generating set chosen for the ideal I. For simplicity, they will be referred to as the $\mathscr{Z}$-complex and the $\mathscr{M}$-complex of the ideal I.

Denote by $\mathscr{Z}_t$ (resp. $\mathscr{B}_t$, $\mathscr{M}_t$) the component in degree t of $\mathscr{Z}$ (resp. $\mathscr{B}$, $\mathscr{M}$) induced by the grading of $\mathscr{S}$. Thus, if we abbreviate $\mathrm{Sym}_s(R^n) = C_s$ (extended by $C_s = 0$ if $s < 0$), $\mathscr{Z}_t$ is the subcomplex

$$0 \to Z_n(\mathscr{K}) \otimes C_{t-n} \xrightarrow{\partial'} \cdots \to Z_1(\mathscr{K}) \otimes C_{t-1} \xrightarrow{\partial'} Z_0(\mathscr{K}) \otimes C_t \to 0.$$

In this notation, $H_0(\mathscr{Z}_t) = \mathrm{Sym}_t(I)$. In the case of $\mathscr{M}$, we shall write

$\mathrm{Sym}_s(S^n) = C_s$ also, as this will not lead to confusion. Note that these subcomplexes have lengths at most n.

The gist of the complex $\mathscr{L}$ is contained in the following:

PROPOSITION 2.1. *For each positive integer t there exists an exact sequence of R-modules*:

$$\cdots \to H_r(\mathscr{Z}_{t+1}) \to H_r(\mathscr{Z}_t) \to H_r(\mathscr{M}_t) \to H_{r-1}(\mathscr{Z}_{t+1}) \to \cdots .$$

Proof. Consider the defining exact sequences

$$0 \to \mathscr{Z}_t \to \mathscr{L}_t \to \mathscr{B}_{t-1}[-1] \to 0$$

and

$$0 \to \mathscr{B}_t \to \mathscr{Z}_t \to \mathscr{M}_t \to 0.$$

As observed earlier, $\mathscr{L}(\partial')$ is the Koszul complexes associated to the indeterminates of the ring $R[T_1,..., T_n]$. Thus

$$\begin{aligned} H_r(\mathscr{L}_t) &= R \qquad \text{for} \quad r = t = 0 \\ &= 0 \qquad \text{otherwise.} \end{aligned}$$

When this is taken into the two long homology sequences, the desired ensues. ∎

Of interest here will be the tail of this exact sequence, that is,

$$\begin{aligned} H_1(\mathscr{M}_t) \xrightarrow{\sigma} H_0(\mathscr{Z}_{t+1}) = \mathrm{Sym}_{t+1}(I) \xrightarrow{\lambda} H_0(\mathscr{Z}_t) = \mathrm{Sym}_t(I) \to H_0(\mathscr{M}_t) \\ = \mathrm{Sym}_t(I/I^2) \to 0, \end{aligned}$$

where we denote by σ (or σ_t) and λ (or λ_t) the connecting homomorphism and the downgrading homomorphism, respectively. Tracing through the meaning of the complex $\mathscr{L}_t$, it is easy to verify—given the exactness of $\mathscr{L}_t$ in positive dimensions—that λ_t is just the map that sends the symmetric power of $t+1$ elements in I, say $b_1 \cdot b_2 \cdot \cdots \cdot b_{t+1}$ into the element $b_1(b_2 \cdot \cdots \cdot b_{t+1})$ of $\mathrm{Sym}_t(I)$. In fact, it does not matter which element is pulled out of the pack. Thus the image of λ_t is the submodule $I(\mathrm{Sym}_t(I))$ of $\mathrm{Sym}_t(I)$.

We point out some consequences of these considerations.

COROLLARY 2.2. *The following are equivalent for the complexes $\mathscr{Z}$ and $\mathscr{M}$.*

(i) *$\mathscr{M}$ is acyclic.*

(ii) *$\mathscr{Z}$ is acyclic and λ is injective.*

Proof. Since $H_r(\mathscr{Z}_0) = 0$ for $r > 0$, the assertions follow by induction and the proposition above. ∎

In particular, this applies to the complexes of [19, 20], where several $\mathscr{M}$-complexes were proved to be acyclic.

We now relate the symmetric algebra of the ideal I, $\mathrm{Sym}(I)$, to $R(I)$, the Rees algebra—or blowing-up ring—of I. $R(I)$ is, we recall, the subring of the polynomial ring $R[x]$ generated by Ix; that is, $R(I)$ is the graded ring whose component in degree t is $I^t x^t$. Thus, $\mathrm{Sym}(I)$ and $R(I)$ coincide in degrees 0 and 1, and, from the universal mapping property of $\mathrm{Sym}(I)$, there exists a canonical homomorphism

$$0 \to \mathscr{A} \to \mathrm{Sym}(I) \to R(I) \to 0.$$

Note that if R is an integral domain, then $\mathscr{A}$ is the torsion submodule of $\mathrm{Sym}(I)$. In all cases, the component of $\mathscr{A}$ in degree 2 coincides with $\delta_1(\mathbf{x})$, for any chosen system of generators for I (cf. [19]).

The relationship between the mapping α and the downgrading mapping λ of the precedent proposition is displayed in the following commutative diagram:

$$\begin{array}{ccc} \mathrm{Sym}_{t+1}(I) & \xrightarrow{\lambda_t} & \mathrm{Sym}_t(I) \\ {\scriptstyle \alpha_{t+1}}\downarrow & & \downarrow{\scriptstyle \alpha_t} \\ I^{t+1} & \xrightarrow[\phi]{} & I^t \end{array}$$

where ϕ is the ordinary inclusion.

Taking into account the exact sequence of (2.1), one has the following statement about the mappings σ, λ and α. It will be our key technical device in checking the equality of the symmetric and Rees algebra of an ideal.

THEOREM 2.3. *Let I be an ideal and let $\mathscr{Z}$ and $\mathscr{M}$ be the complexes constructed on a generating set for I. The following conditions are equivalent*:

(i) *σ is the zero mapping.*

(ii) *λ is injective.*

(iii) *α is an isomorphism.*

Proof. The equivalence of (i) and (ii) being contained in (2.1), we verify the remaining implications. For this we refer to the commutative diagram above: since α_1 and ϕ are injective, an easy induction shows that α_t injective $\Leftrightarrow \lambda_t$ injective for all t. ∎

An immediate consequence of (2.3) is the following criterion of [17] for the equality $\mathrm{Sym}(I) = R(I)$. It does not involve the computations of [17].

COROLLARY 2.4. *Let I be an ideal of the Noetherian ring R. Assume that $I \cap (0 : I) = 0$. Then α: $\mathrm{Sym}(I) \to R(I)$ is an isomorphism if and only if* 0 *is the only element of* $\mathrm{Sym}(I)$ *annihilated by I.*

Proof. Since $H_1(\mathscr{M})$ is annihilated by I, this follows from (2.3). ∎

Remark. This corollary implies that if $\mathrm{Sym}(I)$ and $R(I)$ are isomorphic under some R-homomorphism, then α is an isomorphism.

Before we discuss some applications of (2.2) and (2.3), we quote the main result of [19]—in the improved form of [20]— where the acyclicity of the complex $\mathscr{M}$ was studied.

As a matter of notation, for an R-module M and a prime ideal P, we denote by $\mathrm{depth}(M)_P$ the PR_P-depth of the localization M_P.

THEOREM 2.5. *Let I be an ideal of grade l generated by $n = l + s$ elements. Assume*:

(i) *For each prime ideal $I \subset P$, $l \leqslant \mathrm{height}(P) < l + s$, I_P can be generated by* $\mathrm{height}(P)$ *elements.*

(ii) *For each integer r and each prime ideal $I \subset P$,*

$$\mathrm{depth}(H_r)_P = \inf\{\mathrm{height}(P/I), r\},$$

where the H_r's denote the homology modules of the Koszul complex on a set of n generators for I.

Then the complex $\mathscr{M}$ is acyclic.

The following results puts together some major applications of the present setup.

THEOREM 2.6. *Let R be a Cohen–Macaulay ring and let I be an ideal of grade l generated by $n = l + s$ elements. Assume*:

(i) *For each prime ideal $I \subset P$, $l \leqslant \mathrm{height}(P) < l + s$, I_P can be generated by* $\mathrm{height}(P)$ *elements.*

(ii) *All homology modules of the Koszul complex on a generating set for I are Cohen–Macaulay.*

Then:

(a) *There exist isomorphisms* $\mathrm{Sym}(I) \cong R(I)$ *and* $\mathrm{Sym}(I/I^2) \cong \mathrm{gr}_I(R) =$ *associated graded ring of R.*

(b) $\mathrm{Sym}(I)$ *and* $\mathrm{Sym}(I/I^2)$ *are Cohen–Macaulay rings.*

(c) *Moreover, if* R *is a Gorenstein ring, then* $\mathrm{Sym}(I/I^2)$ *is a Gorenstein ring.*

In the next proof—and on several instances later in Section 5—we shall indulge in a bit of depth-chasing. The context will be the following. Let M be the maximal ideal of the local ring R—or, the irrelevant maximal ideal of a graded ring R—and let

$$0 \to A \to B \to C \to 0$$

be an exact sequence of finitely generated R-modules, resp., graded R-modules. Denote the M-grade of a module by $g(\cdot)$. We have [13, p. 103]:

(a) If $g(B) < g(C)$, then $g(A) = g(B)$.

(b) If $g(B) > g(C)$, then $g(A) = g(C) + 1$.

(c) If $g(B) = g(C)$, then $g(A) \geqslant g(B)$.

We shall be implicitly referring to these depth estimates.

A notational point creeps in the paper from now on: Because the symbols $\mathscr{Z}_t$ and $\mathscr{M}_t$ have been used for the R-subcomplexes of $\mathscr{Z}$ and $\mathscr{M}$ that we have discussed thus far, we shall denote by $(\mathscr{Z})_t$ and $(\mathscr{M})_t$ the components of degree t of $\mathscr{Z}$ and $\mathscr{M}$, but viewed as $R[T_1,..., T_n]$-complexes (cf. Section 1).

Proof of Theorem 2.6. Part (a) follows from Theorem 2.5, Corollary 2.2 and Theorem 2.3.

(b) We may assume that (R, P) is a local ring of dimension, say, d. We may also assume that $l < n = l + s \leqslant d$. Consider the complex $\mathscr{Z}$, which is acyclic by Theorem 2.5 and Corollary 2.2. We estimate the depth of the components of $\mathscr{Z}$ relative to the irrelevant maximal ideal $Q = (P, T_1,..., T_n)$ of $R[T_1,..., T_n]$.

Let $\mathbf{x} = \{x_1,..., x_n\}$ be a generating set for I, and denote by $\mathscr{K} = \mathscr{K}(\mathbf{x})$ the associated ordinary Koszul complex. Note that the complex

$$\begin{aligned} 0 \to K_n \to K_{n-1} \to \cdots \to K_{s+1} \to K_s \to \cdots \\ \to K_1 \to K_0 \to R/I \to 0 \end{aligned}$$

is exact from K_{s+1} upwards. We use it to estimate the Q-depth of the modules

$$(\mathscr{Z})_i = Z_i \otimes R[T_1,..., T_n]$$

in two ranges: (A) $i > s$ and (B) $i \leqslant s$.

(A) $i > s$: Since $B_i = Z_i$, and has as minimal projective resolution

$$0 \to K_n \to \cdots \to K_{i+1} \to B_i \to 0,$$

we have, by the Auslander–Buchsbaum equality [15],

$$P\text{-depth}(Z_i) = d - (n - i - 1) = d + i + 1 - n,$$

and thus Q-depth$(\mathscr{Z})_i = d + i + 1$.

(B) $i \leqslant s$: We may assume that $d \geqslant 2$ as otherwise all B_i, Z_i are Cohen–Macaulay. We begin with $i = 1$; the sequences

$$0 \to Z_1 \to K_1 \to I \to 0,$$
$$0 \to B_1 \to Z_1 \to H_1 \to 0$$

yield the following estimates of the depths of B_1 and Z_1 (in this, we may assume $l \geqslant 2$, as otherwise the exact sequences show that all B_i, Z_i are Cohen–Macaulay): P-depth$(Z_1) = 2 + P$-depth$(R/I) = 2 + d - l$ and P-depth$(B_1) \geqslant 1 + d - l$, since P-depth$(H_1) = d - l$ by hypothesis.

When this estimate of the depth of B_1 is taken into the next exact sequence

$$0 \to Z_2 \to K_2 \to B_1 \to 0$$

we get P-depth$(Z_2) \geqslant 2 + (d - l)$. A simple induction shows that P-depth$(Z_i) \geqslant 2 + (d - l)$, and thus Q-depth$(\mathscr{Z})_i \geqslant 2 + (d - l) + n = 2 + d + s$.

We now show that Sym(I) is Cohen–Macaulay by comparing its dimension to its Q-depth as an $R[T_1, \ldots, T_n]$-module [10].

Since by Theorem 2.3 Sym$(I) \cong R(I)$, dim(Sym(I)) $= d + 1$.

From the exact sequence

$$\begin{tikzcd}[column sep=small] 0 \arrow[r] & (\mathscr{Z})_{n-1} \arrow[rr] \arrow[dr] & & (\mathscr{Z})_{n-2} \arrow[r] & \cdots \\ & & D_{n-2} \arrow[ur] & & \\ \arrow[r] & (\mathscr{Z})_1 \arrow[rr] \arrow[dr] & & (\mathscr{Z})_0 \arrow[r] & \mathrm{Sym}(I) \arrow[r] & 0, \\ & & D_0 \arrow[ur] & & \end{tikzcd}$$

where D_i denotes the image of $(\mathscr{Z})_{i+1}$, and the depth estimates above, we successively arrive at the inequalities Q-depth$(D_{n-1}) \geqslant d + n$, Q-depth$(D_{n-2}) \geqslant d + n - 1$, all the way down to Q-depth(Sym(I)) $\geqslant d + 1$, to conclude the proof.

A similar argument led in [19] to the proof that Sym(I/I^2) is Cohen–Macaulay.

(c) Assume now that R is a Gorenstein ring and consider the $\mathcal{M}$-complex on the chosen n generators of I. Putting $C=(R/I)[T_1,\ldots,T_n]$, we have the exact sequence

$$\begin{aligned}0 &\to H_s \otimes C \to H_{s-1} \otimes C \to \cdots \\ &\to H_1 \otimes C \to C \to \operatorname{Sym}(I/I^2) \to 0.\end{aligned}$$

Note that $\dim(C)=(d-l)+(l+s)=d+s$, while $\dim(\operatorname{Sym}(I/I^2))=d$ (cf. [19]). As H_s is the canonical module of R/I, $K_C=H_s\otimes C$ is the canonical module of C [8]. We must show, according to [8], that $\operatorname{Ext}^s_C(\operatorname{Sym}(I/I^2), K_C)\cong \operatorname{Sym}(I/I^2)$. This follows from the self-duality of this $\mathcal{M}$-complex. Indeed, consider the mapping

$$\psi\colon H_i \to \operatorname{Hom}(H_{s-i}, H_s)$$

defined as $\psi(a)(b)=ab$. Since all H_i are Cohen–Macaulay, the homology algebra $H(\mathcal{K})$ is a Poincaré algebra, that is, the homomorphism ψ is an isomorphism (cf. [7]). ∎

Remarks. (1) Theorem 2.6(c) is reminiscent of a similar result of [10].

(2) $\operatorname{Sym}(I)$, on the other hand, is not always Gorenstein. For instance, if I is generated by a regular sequence of n elements, then $\operatorname{Sym}(I)$ is Gorenstein only when $n\leqslant 2$. In the special case I is generated by the regular sequence $\{x_1,x_2,x_3\}$, $\operatorname{Sym}(I)=R[T_1,T_2,T_3]/J$, where J is generated by the 2×2 minors of the matrix

$$\begin{bmatrix} x_1 & x_2 & x_3 \\ T_1 & T_2 & T_3 \end{bmatrix}.$$

(3) When I is a Cohen–Macaulay ideal of height two in a regular local ring R, then according to the preceding proof, the Z_i's are free. Thus, $\mathcal{Z}$ is a complex of free modules over $R[T_1,\ldots,T_n]$. Therefore, if $\mathcal{Z}$ is acyclic, it provides a free resolution of $\operatorname{Sym}(I)$. In case $\operatorname{Sym}(I)$ has its expected dimension, i.e., $1+\dim(R)$, it will in fact be a complete intersection (cf. [1] or [19]).

Condition (i) in both Theorems 2.5 and 2.6 is necessary for the isomorphism $\operatorname{Sym}(I)\cong R(I)$, even if, as above, we do not have Cohen–Macaulay conditions.

PROPOSITION 2.8. *Let I be an ideal of the Noetherian ring R. If $\operatorname{Sym}(I)\cong R(I)$, then for each prime ideal $I\subset P$, I_P can be generated by height(P) elements.*

Proof. We may assume that (R, P) is a local ring of Krull dimension d. We work with the derived isomorphism $\mathrm{Sym}(I/I^2) \cong \mathrm{gr}_I(R)$.

Let $R'(I)$ be the "large" Rees ring, that is, $R'(I) = R[It, u]$, $u = t^{-1}$; then $\mathrm{gr}_I(R) \cong R'(I)/(u)$. Since $\dim(R'(I)) = d + 1$, for all ideals [18] and u is a regular element, we have $\dim(\mathrm{Sym}(I/I^2)) \leqslant d$. In this case its homomorphic image $\mathrm{Sym}(I/PI)$ has dimension at most d, as desired. ∎

Remark. If above P is a minimal prime of I, then I_P is generated by a system of parameters of R_P. This fact will be discussed again in Proposition 4.1 in the context of determining the Hilbert function of $\mathrm{gr}_I(R)$.

EXAMPLE. One way of constructing ideals with $\mathrm{Sym}(I) = R(I)$, is the following. Let E be a module over the ring A and let R be $\mathrm{Sym}_A(E)$. Pick for I the augmentation ideal of R,

$$I = R_+ = \sum_{t>0} S_t(E).$$

We have the natural R-homomorphism

$$E \otimes_A \mathrm{Sym}_A(E) \to \mathrm{Sym}_A(E)_+ = I \to 0,$$

whose kernel is generated by the elements of the form $e \otimes f \cdot w - f \otimes e \cdot w$, $e, f \in E$, $w \in \mathrm{Sym}_A(E)$. Taking the n-symmetric power of this homomorphism and using the universal property of such powers with respect to the change of rings $A \to R$, we get

$$S_n(E \otimes_A R) \cong S_n(E) \otimes_A R \to S_n(I) \to 0,$$

from which we obtain $I^n \cong S_n(I)$.

In the next section we shall see that this construction does not lead, in general, to acyclic $\mathcal{M}$-complexes.

3. Residual Conditions

In this section, which can be read outside the preceding homological context, we consider residual conditions on the canonical homomorphism α:

$$0 \to \mathcal{A} \to \mathrm{Sym}(I) \xrightarrow{\alpha} R(I) \to 0.$$

It is clear that if an isomorphism, then reduction modulo I gives rise to the isomorphism

$$\alpha^*: \mathrm{Sym}(I/I^2) \cong \mathrm{Sym}(I) \otimes (R/I) \to R(I) \otimes (R/I) = \mathrm{gr}_I(R),$$

where $\mathrm{gr}_I(R) = \sum I^t/I^{t+1}$ is the graded ring associated to the I-adic filtration. Assuming R Noetherian, Valla proved the converse [23].

Here we look at this question but depart slightly from Noetherian conditions. Let I be a finitely generated ideal of the ring R and assume that $R(I)$, the Rees algebra of I, is an R-algebra of finite presentation. This means that we may write $R(I) = R[T_1, ..., T_n]/J$, where J is a finitely generated ideal. It is not difficult to see that in this case whenever we map a polynomial algebra $R[T_1, ..., T_m]$ onto the finitely generated algebra $R(I)$, say by mapping T_i into the element $x_i t$ of $R(I)$, where $\mathbf{x} = \{x_1, ..., x_m\}$ is a generating set for I, the kernel will be finitely generated. If we use this kind of presentation, J is a homogeneous ideal of $R[T_1, ..., T_m]$ and we have $J = (f_1, ..., f_r, f_{r+1}, ..., f_s)$, where the f_i are homogeneous, degree$(f_i) = 1$ for $i \leqslant r$, and degree$(f_i) > 1$ for $i > r$. Thus the R-submodule J_1 generated by $\{f_1, ..., f_r\}$ provides a finite presentation

$$0 \to J_1 \to R_1 \to I \to 0$$

where R_1 is the submodule of 1-forms of $R[T_1, ..., T_m]$. It follows that the algebras $\mathrm{Sym}(I)$ and $\mathrm{Sym}(I/I^2)$ are of finite presentation. Note that $\mathrm{gr}_I(R)$ has finite presentation along with $R(I)$. Furthermore, $\mathscr{A} = \ker(\alpha)$ is a finitely generated ideal of $\mathrm{Sym}(I)$. In particular, each graded component $\mathscr{A}_t$ of $\mathscr{A}$ is a finitely generated R-module.

THEOREM 3.1. *Let I be an ideal of the ring R and assume that the associated Rees algebra $R(I)$ is of finite presentation over R. Then $\alpha\colon \mathrm{Sym}(I) \to R(I)$ is an isomorphism if and only if the reduction $\alpha^*\colon \mathrm{Sym}(I/I^2) \to \mathrm{gr}_I(R)$ is an isomorphism.*

Proof. We may assume that R is a local ring. Consider the diagram associated to α (cf. Theorem 2.3):

$$\begin{tikzcd}
\mathscr{A}_{t+1} \arrow[r] \arrow[d] & \mathscr{A}_t \arrow[d] \\
S_{t+1}(I) \arrow[r, "\lambda"] \arrow[d, "\alpha"'] & S_t(I) \arrow[d, "\alpha"] \\
I^{t+1} \arrow[r, hook] & I^t
\end{tikzcd}$$

By the Snake Lemma, we have the exact sequence

$$0 \to \mathscr{A}_t/\lambda(\mathscr{A}_{t+1}) \to S_t(I/I^2) \to \mathrm{gr}_I(I)_t \to 0.$$

By hypothesis we have $\mathscr{A}_t = \lambda(\mathscr{A}_{t+1})$ for $t \geqslant 2$. On the other hand, since $\mathscr{A}$

is a finitely generated ideal of $\mathrm{Sym}(I)$, there exists an integer $s \geqslant 2$ such that

$$\mathscr{A}_{t+1} = S_1(I) \cdot \mathscr{A}_t, \qquad t \geqslant s.$$

Applying λ to this equation and taking into account the hypothesis above, we get

$$\mathscr{A}_t = \lambda(\mathscr{A}_{t+1}) = \lambda(S_1(I) \cdot \mathscr{A}_t) = I(\mathscr{A}_t),$$

and thus, by Nakayama's lemma, $\mathscr{A}_t = 0$ for $t \geqslant s$. By an obvious descending induction we get $\mathscr{A}_t = 0$ for all t. ∎

Under the same finiteness condition above we have

THEOREM 3.2. *$\mathscr{A} = \ker(\alpha)$ is a nilpotent ideal if and only if $\ker(\alpha^*)$ is nilpotent.*

LEMMA 3.3. *Let I be an ideal of a ring R such that its Rees algebra, $R(I)$, is of finite presentation over R. Then there exists an integer l such that $I^l \cdot \mathscr{A} = 0$.*

Proof. For each prime ideal $I \not\subset P$, the localization α_P is an isomorphism. Thus it follows that for each homogeneous component $\mathscr{A}_t$ of $\mathscr{A}$ the closed set $V(I)$ determined by I contains the support of $\mathscr{A}_t$. Since I and $\mathscr{A}_t$ are both finitely generated, there exists a power of I that annihilates $\mathscr{A}_t$. As $\mathscr{A}$ is a finitely generated ideal of $\mathrm{Sym}(I)$, there exists a power of I annihilating $\mathscr{A}$. ∎

Proof of Theorem 3.2. We only have to show that $\ker(\alpha)$ is nilpotent along with $\ker(\alpha^*)$. For this we refer to the exact sequence in Theorem 3.1. As $\ker(\alpha^*)$ is finitely generated, there exists an integer s such that $[\mathscr{A}_t/\lambda(\mathscr{A}_{t+1})]^s = 0$, that is, $\mathscr{A}_t^s \subset \lambda(\mathscr{A}_{st+1}) \subset I \cdot \mathscr{A}_{ts}$. Now we use the integer l derived from the lemma to get

$$\mathscr{A}_t^{sl} = (\mathscr{A}_t^s)^l \subset I^l \cdot \mathscr{A}_{ts}^l = 0. \quad ∎$$

4. SYZYGETIC IDEALS

In this section we discuss some relationships between the acyclicity of the complex $\mathscr{M}$ associated to an ideal and the notions of "syzygetic" ideal advanced by several authors [16, 19].

The following result was already obtained in Proposition 2.7 via the implied isomorphism between the symmetric and Rees algebras. The proof given here however allows for other consequences. In this section, we are back in Noetherian conditions.

PROPOSITION 4.1. *Let I be an ideal and consider the complex $\mathcal{M}$ associated to a generating set for I. If $\mathcal{M}$ is acyclic, then for every prime ideal P minimal over I, I_P is generated by* height(P) *elements.*

Proof. We may assume that R is a local ring with maximal ideal M and that I is M-primary (cf. [19]). The claim is that I is generated by a system of parameters of R. Assume otherwise, that is, let $\dim(R) = d > n =$ minimum number of generators for I. Pick a minimal generating set of n elements; the corresponding $\mathcal{M}$-complex will be acyclic. For each $t \geqslant 0$, let

$$0 \to H_n \otimes C_{t-n} \to \cdots \to H_1 \otimes C_{t-1} \to C_t \to 0$$

be the subcomplex of $\mathcal{M}$ of degree t (here, $C = S[T_1,\ldots, T_n]$, $S = R/I$). Denote by h_i the length of the S-module H_i. Taking Euler characteristic of this complex, we get

$$\operatorname{length}(H_0(\mathcal{M}_t)) = \operatorname{length}(S_t(I/I^2)) = \sum_{i=0}^{n} (-1)^i h_i \binom{n+t-i-1}{t-i}.$$

The right-hand side is then the Hilbert–Samuel polynomial of the graded algebra $\operatorname{Sym}(I/I^2)$. It is easy to see that the coefficients of t^{n-1} is $\chi(R)/(n-1)!$, where

$$\chi(R) = \sum_{i=0}^{n} (-1)^i h_i$$

is the Euler characteristic of the ordinary Koszul complex associated to the chosen set of generators. Since $n > d$, by [14], $\chi(R) = 0$. Thus $\dim(\operatorname{Sym}(I/I^2)) < n$.

On the other hand, $\operatorname{Sym}_{R/M}(I/MI)$, a homomorphic image of $\operatorname{Sym}(I/I^2)$, has Krull dimension n, since I is minimally generated by n elements. This contradiction proves the assertion. ∎

An interesting application of this setting is the explicit formula of the next corollary. It applies, in particular, to Buchsbaum rings, as we shall see later on.

COROLLARY 4.2. *Let I be generated by a system of parameters of the local ring R, of dimension d. If the associated $\mathcal{M}$-complex is acyclic, then:*

(a) *The Poincaré series of the graded algebra* $\operatorname{gr}_I(R)$ *is given by*

$$P(t) = Q(t)/(1-t)^d, \qquad Q(t) = \sum_{i=0}^{d} (-1)^i h_i t^i,$$

where h_i denotes the length of the homology group H_i of the ordinary Koszul complex associated to a generating set of d elements.

(b) $\operatorname{depth}(\operatorname{gr}_I(R)) \geqslant \operatorname{depth}(R)$.

Proof. (a) Since the acyclicity of $\mathcal{M}$ implies the isomorphism $\operatorname{Sym}(I/I^2) = \operatorname{gr}_I(R)$ (Theorem 2.3), (a) follows from the preceding proof.

Part (b) is a consequence of the argument in Theorem 2.6. Note that $\operatorname{depth}(\operatorname{gr}_I(R))$ is taken relative to its maximal homogeneous ideal. ∎

We now look at the consequences of the acyclicity of an $\mathcal{M}$-complex on so-called syzygetic conditions, specifically the nature of the Koszul homology modules on a set of generators of an ideal I.

Let $\mathcal{K}$ be the Koszul complex associated to a generating set $\mathbf{x} = \{x_1, \ldots, x_n\}$ and denote Z (resp. B) its cycles (resp. boundaries). Consider the homomorphism of graded algebras (cf. Section 2):

$$0 \to \delta(\mathbf{x}) \to H = Z/B \to H^* = Z/Z^* \to 0.$$

The ideal $\delta(\mathbf{x})$ obviously depends on the generating set for I, although, as pointed out in Section 2, its component in degree 2, $\delta_1(\mathbf{x})$, is invariantly defined, being the kernel of the canonial homomorphism $\operatorname{Sym}_2(I) \to I^2$. In fact, over a local ring, it is easy to see that the first non-vanishing $\delta_r(\mathbf{x})$ is independent of the generating set.

The a posteriori role of the vanishing of $\delta(\mathbf{x})$ in the acyclicity of $\mathcal{M}$ will be discussed briefly now. Its a priori role will be examined when we treat specially generated ideals in Section 5.

Among the notions of *syzygetic* ideals used in the past, we single out: (a) I is *syzygetic*$_1$ if $\delta_1(I) = 0$; (b) I is *syzygetic*$_2$ if $\delta(I) = 0$; (c) I is *syzygetic*$_3$ if the $\mathcal{M}$-complex is acyclic.

PROPOSITION 4.3. (c) $\Rightarrow$ (b) $\Rightarrow$ (a).

Proof. We only have to show (c) $\Rightarrow$ (b). Going back to the definition of the approximation complex $\mathcal{M}$, observe that using H^* instead of H gives rise to another complex, $\mathcal{M}^* = H^* \otimes C$; similarly, put $\Delta = \delta(\mathbf{x}) \otimes C$. The exact sequence of complexes

$$0 \to \Delta \to \mathcal{M} \to \mathcal{M}^* \to 0$$

has, in degree t, the following tail:

$$\begin{tikzcd}
0 \arrow[r] & \delta_t(\mathbf{x}) \arrow[r] \arrow[d] & H_t \arrow[r] \arrow[d, "\partial'"] & H_t^* \arrow[r] \arrow[d] & 0 \\
0 \arrow[r] & \delta_{t-1}(\mathbf{x}) \otimes C_1 \arrow[r] & H_{t-1} \otimes C_1 \arrow[r] & H_{t-1}^* \otimes C_1 \arrow[r] & 0
\end{tikzcd}$$

From this diagram, we see that $\delta_1(\mathbf{x}) = H_1(\mathcal{M}_1)$. Assume that we have proved $\delta_{t-1}(\mathbf{x}) = 0$ for $t \geqslant 2$. Since the diagram has exact rows and the middle vertical complex is acyclic, it follows that $\delta_t(\mathbf{x}) = 0$ also. ∎

As for the inverse implications, none is valid:

EXAMPLE 1. Let P be a prime ideal of $R = k[[x, y, z]]$ needing at least five generators; e.g., pick P to be one of the so-called Macaulay primes. According to [2], the homology modules of the Koszul complex $\mathcal{K}$ are Cohen–Macaulay modules, meaning here they are torsion-free R/P-modules. Since P is generically a complete intersection, $(H)_P = (H^*)_P$, and thus $\delta(P) = 0$. We claim that $\mathcal{M}$ is not acyclic. Otherwise it would lead to the equality of the Rees and symmetric algebras of P and this would contradict Proposition 2.7. Thus syzygetic$_2$ does not necessarily imply syzygetic$_3$.

EXAMPLE 2. Let $R = k[u, v, x, y]$, with the relations $ux = vy$, $uy = vx = u^2 = uv = v^2 = 0$. R is a graded ring and may be written (as a k-space): $R = k[x, y] \oplus ku \oplus kux \oplus kv$. Let $I = (x, y)$; note that R is the symmetric algebra of a k-module and I is its augmentation ideal. It follows as in Section 2 that I is syzygetic$_1$. However, I is not syzygetic$_2$ as $Rux = H_1 = Z_2 = \delta_2(I)$.

5. SEQUENCES

In this section we discuss the acyclicity of the complexes $\mathcal{Z}$ and $\mathcal{M}$ for "sequentially" generated ideals, that is, ideals generated by various weakened versions of the notion of regular sequence. In this manner we encounter and strengthen earlier results by several authors, but in the context of the approximation complexes. Besides providing a new family of d-sequences, we prove the acyclicity of the $\mathcal{M}$-complexes for ideals generated by d-sequences, investigate the notion of a singularity locus of the comparison morphism between $\mathrm{Sym}(I)$ and $R(I)$ for ideals generated by proper sequences, and use the $\mathcal{M}$-complex to derive analytic properties of ideals generated by d-sequences. These properties—depth estimates for the modules R/I^t, for large t—were dealt with Huneke [11], but with other proofs. It will also be clear how the notion of proper sequence leads to cases with $\mathrm{Sym}(I)$ Cohen–Macaulay but in weaker conditions than those of Theorem 2.6.

We begin with a partial listing of the definitions of the various kinds of sequences.

DEFINITION. Suppose $\mathbf{x} = \{x_1, ..., x_n\}$ is a sequence of elements in a ring R. The sequence $\mathbf{x}$ is called a:

(a) *d-sequence*, if

(a_1) $\mathbf{x}$ is a minimal generating set of the ideal $(x_1,\ldots,x_n)$;

(a_2) $(x_1,\ldots,x_i): x_{i+1}x_k = (x_1,\ldots,x_i) : x_k$, for $i = 0,\ldots, n-1$ and $k \geqslant i+1$;

(b) *unconditioned d-sequence*, if $\mathbf{x}$, after any reordering is again a d-sequence;

(c) *relative regular sequence*, if $(x_1,\ldots,x_i)I\colon x_{i+1} \cap I = (x_1,\ldots,x_i)$ for $i = 0,\ldots, n-1$, where $I = (x_1,\ldots,x_n)$;

(c*) *relative* regular sequence*, if $(x_1,\ldots,x_i)\colon x_{i+1} \cap I = (x_1,\ldots,x_i)$ for $i = 0,\ldots, n-1$, where $I = (x_1,\ldots,x_n)$;

(d) *proper sequence*, if $x_{i+1}H_j(x_1,\ldots,x_i) = 0$ for $i = 0,\ldots, n-1$, $j > 0$, *where* $H_j(x_1,\ldots,x_i\}$ *denotes the Koszul homology of the sequence* $\{x_1,\ldots,x_i\}$.

The notions (a) and (b) have been developed into a full-fledged theory by Huneke [11, 12]. Both (c) and (c*) were introduced by Fiorentini [5]. As for (d), it is brought in here for its convenience in examining exactness in the complex $\mathscr{M}$. Finally, for purposes of comparison among these various notions, we shall refer to a sequence $\mathbf{x}$ satisfying condition (a_2) above as a *d*-sequence*, that is, a d-sequence possibly stripped of its minimality condition.

Although we shall not deal here with the different strengths of these notions—and there are unconditioned versions for each as well—note that for $n = 1$, (a) through (c*) simply mean that $(0 : x_1) \cap (x_1) = 0$. For $n > 1$, however, these notions differ. Proper sequences are more loosely structured; for instance, the sequence $\{x, y\}$ is proper if and only if $(0 : x) \subseteq (0 : y)$, but it is suitable in studying the complex $\mathscr{Z}$.

Several implications hold between these kind of sequences. We indicate in the diagram below those we shall be interested in. Note that some of these result from the allowed redundancy in definitions (c) through (d).

$$\begin{tikzcd}
 & d\text{-sequence} \arrow[d] & \\
 & d^*\text{-sequence} \arrow[d, leftrightarrow] & \\
 & \text{relative}^* \text{ regular sequence} \arrow[d] & \\
 & \text{relative regular sequence} \arrow[dl] \arrow[dr] & \\
\text{proper sequence} & & \text{syzygetic}_2
\end{tikzcd}$$

5.1.1. Huneke [11] showed the implication: d^*-sequence $\Rightarrow$ relative* regular sequence. For the converse, let $a \in (x_1,\ldots,x_i): x_{i+1}x_k$, $k \geqslant i+1$. Then $ax_k \in (x_1,\ldots,x_i): x_{i+1} \cap I = (x_1,\ldots,x_i)$, that is, $a \in (x_1,\ldots,x_i): x_k$.

5.1.2. Fiorentini [5] actually proved that for a relative regular sequence $\mathbf{x} = \{x_1,\ldots,x_n\}$ one has

$$Z_j(x_1,\ldots,x_i) \cap I\mathscr{K} = B_j(x_1,\ldots,x_i) \qquad \text{for} \quad i = 0,\ldots,n,\ j > 0,$$

where $\mathscr{K}$ denotes the Koszul complex on the sequence $\mathbf{x}$.

For $i = n$ we obtain that the ideal $I = (x_1,\ldots,x_n)$ is syzygetic$_2$ (cf. Section 4). It is also clear that the equality above implies that $\mathbf{x}$ is proper.

5.1.3. Many classes of d-sequences are known (see [5, 11, 12, 21]). Recently Huneke communicated to us that the ideals of Theorem 2.6 can (locally) be generated by d-sequences.

We now add another family of examples, that of monomial ideals. Let $\{T_1,\ldots,T_m\}$ be a regular sequence in the ring R, and denote by $\mathbf{x} = \{x_1,\ldots,x_n\}$ a sequence of "monomials" in the T_i's. We shall say that it is a minimal sequence if none of its monomials is a non-trivial multiple of another. This is equivalent to saying that $\mathbf{x}$ is a minimal generating set for the ideal $(x_1,\ldots,x_n)$.

In the sequel, we shall use the notation $[x, y]$ to denote the greatest common divisor of the monomials x and y; $x \mid y$ denotes that x divides y.

PROPOSITION 5.2. *Let* $\mathbf{x} = \{x_1,\ldots,x_n\}$ *be a minimal sequence of monomials. Assume that* $[x_i, x_j] = [x_i, x_j^2]$, *for* $i < j$. *The following conditions are equivalent*:

(a) *For all* $i < j < k$, *we have* $[x_i, x_j] \mid x_k$.

(b) $\mathbf{x}$ *is a* d-*sequence.*

The hypothesis $[x_i, x_j] = [x_i, x_j^2]$ means that whenever a "variable" T_i appears in the sequence, it occurs to non-decreasing powers. Thus the sequence $\{T_1T_3, T_2^2, T_1T_2\}$ is not a d-sequence, although (a) above is satisfied.

For the proof of Proposition 5.2 we shall need the following fact from the theory of Taylor's resolutions ([22]; see also [6]).

LEMMA. *If* $x_1,\ldots,x_n$, y *are monomials, then* $(x_1,\ldots,x_n): y = (x_1/[y, x_1],\ldots,x_n/[y, x_n])$.

A key consequence we shall use in the proof of Proposition 5.2 is:

(a) If $y \in (x_1, \dots, x_n)$, then $y = ax_i$ for some i. In particular, a minimal system of monomials of $(x_1, \dots, x_n)$ is, up to order, uniquely determined.

(b) Finally, note that if there is a relation $ax = by$, x, y monomials, then $a = a' \cdot y/[y, x]$.

Proof of Proposition 5.2. We first prove (a) $\Rightarrow$ (b). We check directly condition (a_2) in the definition of d-sequences.

For $i < j \leqslant k$ we have $[x_i, x_j x_k] = [x_i, x_j]$ from the overall assumption if $j = k$, and $[x_i, x_j x_k] = [x_i, x_k]$ from condition (a) if $j < k$. When these are taken into the lemma, one gets (b).

To show (b) $\Rightarrow$ (a) we use that d-sequences are relative* regular sequences (cf. 5.1.1).

We may assume $n > 2$. We show by induction on i the following statement:

(#) For all $j < l \leqslant i$ and $k > i$ we have $[x_j, x_l] \mid x_k$.

$i = 2$: We have to show that $[x_1, x_2] \mid x_k$ for $k \geqslant 3$. Note that

$$a_k = x_1 \cdot x_2/[x_1, x_2] \cdot [x_1/[x_1, x_2], x_k] \in (x_1): x_2 \cap I \qquad \text{for all } k \geqslant 3.$$

Since $[x_1, x_2]$ and $[x_1/[x_1, x_2], x_k]$ have no common divisor, it follows that $a_k \in (x_1)$ if and only if $[x_1, x_2] \mid x_k$.

$i > 2$: We show that $[x_j, x_{i+1}] \mid x_k$ for $j \leqslant i$, $k > i + 1$. We have

$$b = x_j x_k/[x_j, x_{i+1}] \cdot [x_j/[x_j, x_{i+1}], x_k] \in (x_1, \dots, x_i): x_{i+1} \cap I.$$

Hence if $\mathbf{x}$ is a relative* regular sequence there exists $l \leqslant i$ such that $b = ax_l$ for some $a \in R$. This implies

$$x_k/[x_j/[x_j, x_{i+1}], x_k] = a' \cdot x_l/[x_j/[x_j, x_{i+1}], x_l]$$

and hence

$$(\#\#) \quad x_k = a' \cdot x_l[x_j/[x_j, x_{i+1}], x_k]/[x_j/[x_j, x_{i+1}], x_l].$$

Assume $l \neq j$; then $[x_j, x_l] \mid x_k$ by the induction hypothesis, and thus $[x_j, x_l] \mid [x_j, x_k]$ and $[x_j/[x_j, x_{i+1}], x_l] \mid [x_j/[x_j, x_{i+1}], x_k]$. But then (##) implies $x_l \mid x_k$, a contradiction.

Hence we must have $l = j$ and, in this case, (##) implies $x_k = a'[x_j/[x_j, x_{i+1}], x_k] \cdot [x_j, x_{i+1}]$, and again we get $[x_j, x_{i+1}] \mid x_k$. ∎

COROLLARY 5.3. *If $\mathbf{x} = \{x_1, \dots, x_n\}$ is a sequence of distinct square-free monomials, then $\mathbf{x}$ is a d-sequence if and only if $[x_i, x_j] \mid x_k$, for $i < j < k$.*

We now look at the import of these sequences on the approximation complexes.

THEOREM 5.4. *Let* $\mathbf{x}$ *be a proper sequence. Then the corresponding* $\mathscr{Z}$*-complex is acyclic.*

Proof. We let $\{e_1,..., e_n\}$ denote the standard basis of $\mathscr{K}_1 = R^n$; we will also let it stand for the basis of $S_1(R^n)$ = first symmetric power of R^n (cf. Section 1).

Let $z \in \mathscr{Z}$; since we view the elements of $\mathscr{Z}$ as polynomials in $e_1,..., e_n$ with coefficients in $Z(\mathscr{K})$, we can write

$$z = \sum z_{h,i} \otimes he_j^i$$

where the h's are distinct monomials in $e_1,..., e_{j-1}$, and $\partial(z_{h,i}) = 0$ for all h, i. We prove by induction on j that $z = \partial'(b)$, with $b \in \mathscr{Z}$.

$j = 1$: We write simply

$$z = \sum z_i \otimes e_1^i.$$

We claim that $z_i \in Z(x_1)$ for all i. Indeed, let $k = \inf\{l \mid z_i \in Z(x_1,..., x_l)\}$. Assume $k > 1$. Each z_i can then be written as

$$z_i = z_i' + w_i' \wedge e_k \qquad \text{with } z_i',\ w_i' \in \mathscr{K}(x_1,..., x_{k-1}).$$

Since

$$0 = \partial'(z) = \sum w_i' \otimes e_1^i e_k + (\text{terms in } e_1,..., e_{k-1}),$$

it follows that $w_i' = 0$. Thus $z_i \in \mathscr{K}(x_1,..., x_{k-1})$, a contradiction.

We now have

$$z = \sum a_i e_1 \otimes e_1^i, \qquad a_i x_1 = 0.$$

Since $0 = \partial'(z) = \sum a_i e_1^{i+1}$, it follows that $z = 0$.

$j > 1$: As in the preceding case, it can be shown that $z_{h,i} \in Z(x_1,..., x_j)$. We now prove by induction on $\deg_{e_j}(z) = t$ that $z = \partial'(b)$.

If $t = 0$, then by our induction hypothesis on j, we are done. Suppose then $t > 0$. We write

$$z_{h,t} = z_{h,t}' + w_{h,t}' \wedge e_j$$

as before (if $z_{h,t} \in Z_1(\mathscr{K})$, then $z_{h,t} = z_{h,t}' + w_{h,t}' e_j$, $w_{h,t}' \in R$). Then

$$0 = \partial'(z) = \sum_h w_{h,t}' \otimes he_j^{t+1} + (\text{terms of lower } e_j\text{-degree}).$$

It follows that all $w_{h,t}' = 0$ and hence $z_{h,t} \in Z(x_1,..., x_{j-1})$.

On the other hand, we have

$$\partial(z_{h,t} \wedge e_j \otimes he_j^{t-1}) = x_j z_{h,t} \otimes he_j^{t-1}.$$

Since $\mathbf{x}$ is proper, there exists $b_{h,t} \in \mathscr{K}(x_1,..., x_{j-1})$ with $\partial(b_{h,t}) = x_j z_{h,t}$. Let

$$b = \sum_h (z_{h,t} \wedge e_j \otimes he_j^{t-1} - b_{h,t} \otimes he_j^{t-1}).$$

Then $b \in \mathscr{Z}$ and $\deg_{e_j}(z - \partial'(b)) < t$. By induction, $z - \partial'(b)$ is a boundary and the proof is complete. ∎

COROLLARY 5.5. *Let I be an ideal generated by a proper sequence, and consider the associated $\mathscr{M}$-complex. The following conditions are equivalent*:

(a) *$\mathscr{M}$ is acyclic.*

(b) $H_1(\mathscr{M}) = 0$.

(c) $\mathrm{Sym}(I) \cong R(I)$.

(d) $\mathrm{Sym}(I/I^2) \cong \mathrm{gr}_I(R)$.

Proof. Follows from the above, Corollary 2.2, and Theorems 2.3 and 3.1. ∎

THEOREM 5.6. *Let $\mathbf{x}$ be a relative* regular sequence. Then the corresponding $\mathscr{M}$-complex is acyclic.*

Proof. By the preceding theorem it is enough to show that $H_1(\mathscr{M}) = 0$. Let z represent a 1-cycle of $\mathscr{M}$. We write

$$z = \sum z_{h,i} \otimes he_j^i,$$

the h's the distinct monomials in $e_1,..., e_{j-1}$, $z_{h,i} \in Z_1(\mathscr{K})$, $z_{h,i} = \sum_{k=1}^n \alpha_{h,i}^k e_k$.

We proceed by induction on j.

$j = 1$: Then $z = \sum z_i \otimes e_1^i$, $z_i = \sum \alpha_i^k e_k$. Since $\partial'(z) = \sum \alpha_i^1 e_1^{i+1} + \sum_{k>1} \alpha_i^k e_1^i \cdot e_k \in I\mathscr{L}$ (cf. Section 1), it follows that all $z_i \in I\mathscr{K}$. Since x is a relative regular sequence, by 5.1.2 we have that all z_i are boundaries, so z is also a boundary.

$j > 1$: Since $\partial'(z) \in I\mathscr{L}$, it follows that $\alpha_{h,i}^k \in I$ for all h, i and $k \geqslant j$. Hence $\alpha_{h,i}^n \in (x_1,..., x_{n-1})$: $x_n \cap I = (x_1,..., x_{n-1})$. Thus we can find $b_{h,i} \in \mathscr{K}(x_1,..., x_{n-1})$ with $\partial(b_{h,i}) = \alpha_{h,i}^n$. In this case $z_{h,i} + \partial(b_{h,i} \wedge e_n) = \sum_{k=1}^{n-1} \beta_{h,i}^k e_k$. By induction we may therefore assume that all $z_{h,i} \in \mathscr{K}(x_1,..., x_j)$.

We now proceed by induction on $\deg_{e_j}(z) = t$. We may assume $t > 0$. Then $\beta^j_{h,t} \in I$ for all h and also $\beta^j_{j,t} \in (x_1,\ldots,x_{j-1}) : x_j$; thus $\beta^j_{h,t} \in (x_1,\ldots,x_{j-1})$. Pick $b_{h,t} \in \mathscr{K}(x_1,\ldots,x_{j-1})$ with $\partial(b_{h,t}) = \beta^i_{h,t}$. It follows that

$$z_{h,t} + \partial(b_{h,t} \wedge e_j) = \sum_{k=1}^{j-1} \gamma^k_{h,t} e_k.$$

Now we use that $\mathbf{x}$ is a proper sequence and the rest of the proof is the same as that of Theorem 5.4. ∎

Remark. The equality of the symmetric and Rees algebras of an ideal generated by a d-sequence now follows from Corollary 5.5. For ideals generated by unconditioned d-sequences this had been proved earlier by Huneke [12] and Valla [23]. (For d-sequences this result also appears in a preprint form of [2].)

Note also that for local rings the length of a d-sequence is, by Theorem 5.6 and Proposition 2.7, bounded by the dimension of the ring.

COROLLARY 5.7. *If* $\mathbf{x}$ *is a system of parameters of a local Buchsbaum ring, then the corresponding* $\mathscr{M}$*-complex is acyclic.*

Proof. For Buchsbaum rings (cf. [21]), the system of parameters are d-sequences [11, 1.8]. ∎

As a consequence, Corollary 4.2 provides considerable information about the depth and the multiplicities of the ring $\mathrm{gr}_{(\mathbf{x})}(R)$.

We consider now some applications to ideals generated by proper sequences with high "residual" depths. First, observe that for a proper sequence $\mathbf{x} = \{x_1,\ldots,x_n\}$ one has the following exact sequences:

$$0 \to H_1(x_1) \to R \xrightarrow{x_i} R \to R/(x_1) \to 0,$$

$$0 \to H_1(x_1,\ldots,x_i) \to H_1(x_1,\ldots,x_{i+1})$$

$$\to R_i \xrightarrow{x_{i+1}} R_i \to R_{i+1} \to 0,$$

where $R_i = R/(x_1,\ldots,x_i)$, and, for $j > 1$,

$$0 \to H_j(x_1,\ldots,x_i) \to H_j(x_1,\ldots,x_{i+1}) \to H_{j-1}(x_1,\ldots,x_i) \to 0,$$

derived from the definition and the inductive sequence for the ordinary Koszul homology [15].

From these sequences one can easily obtain depth estimates for $H_j(x_1,\ldots,x_n)$ in terms of $\mathrm{depth}(R/(x_1,\ldots,x_i))$.

PROPOSITION 5.8. *Let* $\mathbf{x} = \{x_1,..., x_n\}$ *be a proper sequence of the local ring* R. *Assume that* $\operatorname{depth}(R/(x_1,..., x_i)) \geqslant d - i$, $d = \operatorname{depth}(R)$. *Then*

(a) $\operatorname{depth} H_j(x_1,..., x_n) \geqslant d - (n - j)$ *for* $j = 0,..., n$.

For the corresponding $\mathscr{M}$*-complex, we have*

(b₁) $\operatorname{depth} H_1(\mathscr{M}_t) \geqslant 1$ *if* $d \geqslant n$, $t \geqslant 0$;

(b₂) $\operatorname{depth} H_1(\mathscr{M}_t) \geqslant 2$ *if* $d > n$, $t \geqslant 0$.

Proof. Part (a) follows simply from the depth-chasing arguments we have used so often, as applicable to the exact sequences above.

(b) From Theorem 5.4 and Corollary 2.2, the complexes $\mathscr{M}_t$ are exact in dimensions two and higher. Consider one of them:

$$0 \to H_n \otimes C_{t-n} \to \cdots \to H_2 \otimes C_{t-2}$$
$$\to H_1 \otimes C_{t-1} \to C_t \to 0.$$

Since the complex is exact from $H_2 \otimes C_{t-2}$ upwards, the depth estimates of (a) imply that the module of 1-boundaries, B, has depth at least $\operatorname{depth}(R) - n + 2$. On the other hand, from the exact sequence for the 1-cycles

$$0 \to Z \to H_1 \otimes C_{t-1} \to C_t$$

we have that $\operatorname{depth}(Z)$ is at least 1 if $d \geqslant n$, and at least 2 if $d > n$. When this is taken into the sequence

$$0 \to B \to Z \to H_1(\mathscr{M}_t) \to 0,$$

we get the stated results. ∎

This proposition gives a measure of the "singularity locus" of the homomorphism $\alpha: \operatorname{Sym}(I) \to R(I)$. Indeed, let I be generated by a proper sequence and consider the diagram (cf. Theorem 3.1):

$$\begin{tikzcd} & \mathscr{A}_{t+1} \arrow[r] \arrow[d] & \mathscr{A}_t \arrow[d] \\ 0 \arrow[r] & H_1(\mathscr{M}_t) \arrow[r] & S_{t+1}(I) \arrow[r, "\lambda"] \arrow[d, "\alpha"] & S_t(I) \arrow[d, "\alpha"] \\ & & I^{t+1} \arrow[r] & I^t. \end{tikzcd}$$

COROLLARY 5.9. *Let I be an ideal generated by a proper sequence as in Proposition* 5.8. *Then α is an isomorphism in the following two cases*:

(b_1) depth$(R) \geqslant n$ *and* α *is an isomorphism on the punctured spectrum of* R.

(b_2) depth$(R) > n$ *and* α *is an isomorphism at all primes of dimension at least* 1.

Proof. Since $H_1(\mathcal{M}_t)$ embeds in $\mathcal{A}_{t+1}$, for each t, and the hypothesis on α means that $\dim(\mathcal{A}_{t+1}) = 0$ in case (b_1), or $\dim(\mathcal{A}_{t+1}) \leqslant 1$ in case (b_2), the assertions follow from Proposition 5.8. ∎

Remark. One instance where the residual depth condition are satisfied is that of monomial ideals. In fact, if $I = (x_1, ..., x_n)$, x_i = monomial, it follows from [22] that the projective dimension of $R/(x_1, ..., x_i)$ is at most i. Thus, by the Auslander–Buchsbaum equality [15], one has that depth$(R/(x_1, ..., x_i))$ is, locally, at least depth$(R) - i$.

COROLLARY 5.10. *Let R be a local ring and let I be a non-zero ideal generated by the proper sequence* $\mathbf{x} = \{x_1, ..., x_n\}$ *satisfying the residual grade conditions of Proposition* 5.8. *Then* depth(Sym$(I)) \geqslant$ depth$(R) + 1$. (depth(Sym(I)) = *grade of its irrelevant maximal ideal.*)

Proof. We proceed as in Theorem 2.6(b): Since the $\mathcal{Z}$-complex is acyclic by Theorem 5.4, we estimate the depth of its component in degree i:

$$(\mathcal{Z})_i = Z_i(\mathcal{K}) \otimes R[T_1, ..., T_n].$$

To start the depth-chasing of the modules Z_i, note that $Z_0 = R$, and from the exact sequence

$$0 \to Z_1 \to R^n \to R \to R/I \to 0$$

we get depth$(Z_1) \geqslant$ depth$(R) - n + 2$. Taken into the sequence

$$0 \to B_1 \to Z_1 \to H_1 \to 0,$$

and using the estimate of Proposition 5.8, we get depth$(B_1) \geqslant$ depth$(R) - (n-1)$; this allows another round of estimates. It is clear that one obtains depth$(Z_i) \geqslant$ depth$(R) - (n-i) + 1$, for $i \geqslant 1$. The rest of the proof goes as in Theorem 2.6(b). ∎

COROLLARY 5.11. *Let R be a Cohen–Macaulay ring and let I be an ideal generated by a sequence of monomials* $\mathbf{x} = \{x_1, ..., x_n\}$ *satisfying the conditions*:

(a) $[x_i, x_j] \mid x_k$ *for* $i < j < k$;

(b) $[x_i, x_j] = [x_i, x_j^2]$ *for* $i < j$.

Then Sym(I) *is a Cohen–Macaulay ring.*

Proof. By Proposition 5.2, I is generated by a d-sequence; thus, if $n \neq 0$, $\dim(\mathrm{Sym}(I)) = \dim(R) + 1$, since $\mathrm{Sym}(I) \cong R(I)$. On the other hand, as remarked earlier, the residual depth conditions of Proposition 5.8 are always satisfies for monomial ideals; the assertion now follows from Corollary 5.10. ∎

The next application is another proof of a theorem of Huneke [11]. We recast it in a slightly different form.

THEOREM 5.12. *Let (R, M) be a Cohen–Macaulay local ring and let I be an ideal minimally generated by the proper sequence $\mathbf{x} = \{x_1, \ldots, x_n\}$. Assume that*: (a) $\mathrm{height}(I) > 0$; (b) $\mathrm{Sym}(I) \cong R(I)$; (c) $\mathrm{depth}(R/(x_1, \ldots, x_i)) \geqslant \mathrm{depth}(R) - i$, $i = 1, \ldots, n$.

Then $n = \dim(R) - \inf_t\{\mathrm{depth}(R/I^t)\}$.

Proof. Note that for each t the complex $\mathscr{M}_t$ is acyclic. Taking into account the depth estimates for the Koszul homology modules provided by Proposition 5.8, we have, as in Corollary 5.10, that

$$n \geqslant \dim(R) - \inf_t\{\mathrm{depth}(I^t/I^{t+1})\}.$$

We recall a formula of Burch [4] on the analytic spread $l(I)$ of the ideal I. ($l(I)$ is the Krull dimension of the special fibre of $R(I)$, that is, $\dim(R(I) \otimes (R/M))$, which, in our case, is just n.) The formula states that

$$l(I) \leqslant \dim(R) - \inf_t\{\mathrm{depth}(R/I^t)\}.$$

Brodmann [3], on the other hand, has shown that the infimum above can be replaced by its asymptotic value—which exists in much broader situations—and furthermore, under Cohen–Macaulay conditions (and $\mathrm{height}(I) > 0$), one has the equality of asymptotic values

$$\mathrm{asym}\ \inf_t\{\mathrm{depth}(I^t/I^{t+1})\} = \mathrm{asym}\ \inf_t\{\mathrm{depth}(R/I^t)\}.$$

Altogether we have the desired equality. ∎

Finally, we elucidate a syzygetic condition of a sporadic nature, the so-called condition (γ) of [17].

We shall say that an ideal I satisfies the (γ)-condition if its admits a generating set $\mathbf{x} = \{x_1, \ldots, x_n\}$ with the property that whenever $\sum r_i x_i = 0$ and $r_n \in I$, then all $r_i \in I$. On the other hand, recall that an ideal I is said to be an almost complete intersection if it can be generated by $\mathrm{grade}(I) + 1$ elements.

THEOREM 5.13. *Let I be an ideal of the local ring R. If I satisfies the (γ)-condition and $\delta_1(I) = 0$, then*:

(a) *I is an almost complete intersection*;

(b) *I is generated by a d-sequence.*

Proof. Assume that $\mathbf{x} = \{x_1,..., x_n\}$ is a generating set for which (γ) holds. Pick a basis $e_1,..., e_{n-1}$, e of R^n and map $e_i \to x_i$, $e \to x_n = x$; for simplicity write $R^n = R^{n-1} + Re$. Denote by Z_1 (resp. B_1) the module of 1-cycles (resp. 1-boundaries under this presentation, and by $Z_1(J)$ those corresponding to the ideal $J = (x_1,..., x_{n-1})$.

The (γ)-condition means that $Z_1 \cap (R^{n-1} + Ie) = Z_1 \cap IR^n$, and, in particular, implies $Z_1(J) \subset IR^n$. The other condition, $\delta_1(I) = 0$, means $Z_1 \cap IR^n = B_1$. We thus have $Z_1(J) \subset B_1$.

(a) Consider the recursive exact sequence for the ordinary Koszul homology [15]:

$$0 \to H_1(J)/xH_1(J) \to H_1(I) \to (J : x)/J \to 0.$$

Since $Z_1(J) \subset B_1$, the inclusion at the left is the trivial map, and thus $H_1(J) = 0$ by Nakayama's lemma. This shows that $\{x_1,..., x_{n-1}\}$ form a regular sequence [15], and grade$(I) \geqslant n - 1$, as desired.

(b) If I is minimally generated by $n - 1$ elements, we are done; so assume $\mathbf{x}$ is a minimal generating for I. We verify that $\mathbf{x}$ is a relative* regular sequence. Since the first $n - 1$ elements form a regular sequence, we only have to check that $(J : x) \cap I = J$. For this consider the diagram induced by the projection of Z_1 onto its last "coordinate":

$$\begin{array}{ccccccccc}
0 & \longrightarrow & Z_1(J) & \longrightarrow & B_1 = Z_1 \cap IR^n & \longrightarrow & (J : x) \cap I & \longrightarrow & 0 \\
 & & \big\downarrow{=} & & \big\downarrow & & \big\downarrow & & \\
0 & \longrightarrow & Z_1(J) & \longrightarrow & Z_1 & \longrightarrow & J : x & \longrightarrow & 0.
\end{array}$$

Thus $H_1(I) \cong (J : x)/(J : x) \cap I$; since also $H_1(I) \cong (J : x)/J$, we easily get $(J : x) \cap I = J$. ∎

Acknowledgments

Parts of this work were carried out at the VI Escola de Algebra of the Brazilian Mathematical Society, at Recife, Brazil. We are grateful to the Universidade Federal de Pernambuco (UFPe) and the Brazilian Research Council (CNPq) for their hospitality and support. J.H. was sponsored by the GMD–CNPq Exchange Program, A.S. was on leave at UFPe, and W.V.V. was partially supported by CNPq and a NSF grant.

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